20200127

Random Variables

Let (S,P) be a sample space. A **random variable** is a function X from S to V where V is some set.

Usually V is a set of numbers, and X is a function that measures some attribute of the outcome.

Example: Let S = {result of tossing a red die and a blue die} = {(1,1), ..., (6,6)}

Let X(s) = the sum of the two values in s

So X((3,2)) = 5, X((4,6)) = 10, etc.

Another Example: Let S = {result of tossing a coin 5 times} = {HHHHH, HHHHT, ..., TTTTT}

Let X(s) = the number of Heads in s

So X(HHHHH) = 5, X(THHTT) = 2, etc.

Yet Another Example: Let S = {10,11,12,13,14,15} and let P(i) = 1/6 $\forall i \in S$

Let X(s) = s % 4 (where % means "mod")

s	X(s)
10	2
11	3
12	0
13	1
14	2
15	3

Let's pursue this example a bit further. We can define events based on X. For example, "X = 2" is an event, as is "X $\in$ {1,2}", "X $\geq$ 2", etc. And since we have events, it makes sense to determine the probability of these events. So what is P(X = 2)?

Looking at the table above, we can see that X(s) = 2 occurs when s $\in$ {10,14}, and since 10 and 14 have combined probabilities = $\frac{1}{3}$, we can see that $P(X = 2) = \frac{1}{3}$

Now let's look at the coin flipping example described previously:

Let S = {result of tossing a coin 5 times} = {HHHHH, HHHHT, ..., TTTTT}

Let X(s) = the number of Heads in s

So X(HHHHH) = 5, X(THHTT) = 2, etc.

Now we can ask questions such as "What is the probability that X = 2?" or equivalently, "What is P(X = 2)?"

At this point, many people (who should know better) make a crucial mistake: they assume without any justification that the coin we are flipping is balanced (i.e. H and T have equal probability of coming up on each toss). We can't make that assumption, so we deal with the more general case. Suppose the coin has probability p of coming up H, and therefore probability (1-p) of coming up T.

What is P(X = 2) with X(s) defined as the number of Heads in s?

The coin tosses are independent (the coin has no memory), so the probability of tossing the outcome HHTTT (2 heads and then 3 tails) is

$$p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p)$$

This is one of the outcomes that gives X = 2

But so is TTHTH, which has probability $(1-p) \cdot (1-p) \cdot p \cdot (1-p) \cdot p$

and in fact we can see that every outcome containing exactly 2 H's will have probability $p^2 \cdot (1-p)^3$, and P(X=2) will just be the sum of all of these. How many are there?

The two heads can occur in any of the five positions, so a simple counting argument tells us there are $\binom{5}{2}$ different outcomes with exactly 2 H's. The final answer is:

$$P(X=2) = \binom{5}{2} \cdot p^2 \cdot (1-p)^3$$

Similarly we can compute $P(X=4) = \binom{5}{4} \cdot p^4 \cdot (1-p)$

And of course this generalizes completely: when we toss the coin n times, the probability that X = k is given by

$$P(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

which we all know and love as the Binomial Theorem or Binomial Formula. What's it doing here in our discussion of probability? It's showing us (again!) that probability theory is a branch of discrete math.

Exercise: Suppose we are tossing a coin with P(H) = 1/3, P(T) = 2/3. If we toss the coin four times and X(s) is the number of Heads we see in outcome s, what is P(X=2)?