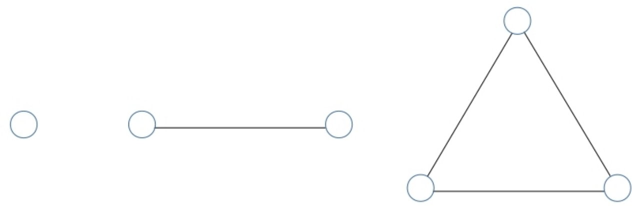
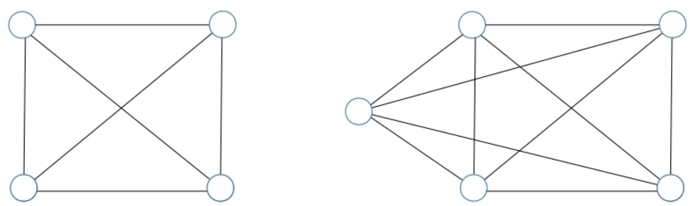
Cliques, Independent Sets and Complements

We call a graph **complete** if it contains all possible edges. The notation for the complete graph on n vertices is K_n



K1 K2 K3



K4 K5

If H is a subgraph of G and H is a complete graph, we say H is a **k-clique** of G. (Some people use the term k-clique as a synonym for K_n --- that's ok.)

Side note: an innocent-sounding question is "Given a graph G, what is the **largest** k-clique that is a subgraph of G?" Simple as it sounds, this question is so difficult that nobody has ever discovered a good way to find the answer, despite decades of research. This problem actually falls into the same category of problems as "Given a partial order P, what is the dimension of P?" and "Given a graph G, does G have a Hamilton Cycle?" : they are all among the hardest problems in computer science, and they are all equally difficult.

Why is this problem even interesting? It turns out that it is relevant to lots of very practical problems. For example, suppose the vertices of the graph represent Queen's courses, and two vertices are joined by an edge if and only if there is at least one student taking both courses. When it comes time to schedule final exams, any vertices that are adjacent must be scheduled into different time-slots. So if there is a k-clique, the exam schedule will need at least k time-slots. Knowing the size of the largest k-clique would give a lower bound on the number of time-slots needed.

A subgraph with k vertices and **no edges** is called a **k-independent set**. K-independent sets are also useful in applications such as the exam-scheduling problem we just looked at. If we can find the largest independent set in G, those vertices represent a set of courses that can all be scheduled into the same exam slot.

Unfortunately, finding the largest independent set in a graph G is **just as hard** as finding the largest k-clique in G.

K-cliques and k-independent sets are a bit like chains and anti-chains in a poset: a k-clique is a set of vertices that are all related, and a k-independent set is a set of vertices that are all unrelated.

Complement

Let $G = (V, E)$ be a graph. We define the **complement** of G as $\bar{G} = (V, \bar{E})$ where $\bar{E}$ contains all the edges that are not in E. As an exercise, determine the complement of some of the graphs in these notes.

It should be easy to see that the complement of a k-clique is a k-independent set and vice versa, and that H is a k-clique in G if and only if $\bar{H}$ is a k-independent set in $\bar{G}$

This means that if we could find the largest k-clique in all graphs, we could use this to find the largest independent set as follows:

to find the largest independent set in G,
- construct $\bar{G}$
- find the largest k-clique in $\bar{G}$
- these vertices form the largest independent set in G

You should convince yourself that this works in reverse also: if we could find the largest independent set in all graphs, we could use this to find the largest clique in another graph.

This idea – two different-seeming problems that turn out to be closely related – will be crucial in CISC-365.