CISC-365*
Test #1
January 30, 2019

Student Number (Required) ______________________

Name (Optional)________________________________

This is a closed book test. You may not refer to any resources.

This is a 50 minute test.

Please write your answers in ink. Pencil answers will be marked, **but will not be re-marked under any circumstances.**

The test will be marked out of 50.

Question 1	/15
Question 2	/10
Question 3	/10
Question 4 or 5	/15
TOTAL	/50

QUESTION 1 (15 Marks)

Let X be a problem in the NP class. The details of X are unimportant but you can assume that each instance of X consists of a set of n integers, and another integer t.

(Parts (a) through (e) are independent of each other. Each part is worth 3 marks)

(a) Suppose we find an algorithm that solves X in $O(2^n)$ time. Does this give us any information about whether X is in P, or whether X is NP-Complete? Explain.

(b) Suppose we are able to prove that every possible algorithm for X requires at least 2^n steps. Does this give us any information about the classes P, NP, and NP-Complete? Explain.

(c) Suppose we are able to show that $X \propto k\text{-}Clique$. Does this give us any information about whether X is in P, or whether X is NP-Complete? Explain.

(d) Suppose we are able to show that $k\text{-}Clique \propto X$. Does this give us any information about whether X is in P, or whether X is NP-Complete? Explain.

(e) Suppose we find an algorithm that solves X in $O(n^t)$ time (remember that t is part of the instance definition). Does this give us any information about whether X is in P, or whether X is NP-Complete? Explain.

QUESTION 2 (10 Marks)

The 3-Colouring Problem 3COL: Given a graph G on n vertices, can we colour the vertices of G using no more than 3 colours in such a way that no vertices that are joined by an edge have the same colour?

The 2-Colouring Problem 2COL: Given a graph G on n vertices, can we colour the vertices of G using no more than 2 colours in such a way that no vertices that are joined by an edge have the same colour?

3COL is known to be NP-Complete. However there is a polynomial-time algorithm for 2COL. We can call this algorithm 2C-ALG.

Consider this algorithm for 3COL:

```
# Let the colours be red, yellow, blue
For each subset T of the vertex set of G: {
     if T contains any vertices that are adjacent:
          skip this T
     else:
          colour all vertices in T red
          temporarily delete these vertices from G
          use the polynomial-time 2C-ALG algorithm to see if
               the remaining vertices can be properly
               coloured with yellow and blue
          if the answer is "Yes": print "Yes" and exit
          else: restore G to its original state
}
print "No"     # all attempts to 3-colour G have failed
```

This algorithm correctly solves 3COL .

Does this algorithm prove P = NP? Explain why or why not. If this space is too small for your answer, please use the back of this page.

QUESTION 3 (10 marks)

Consider this variant of the Subset Sum problem:

25_Value_Subset_Sum: Given a set S of exactly 25 integers and a target integer k, does S contain a subset that sums to k?

Prove this problem is in **P** by describing an algorithm to solve any instance of the problem in polynomial time. You are not required to express your algorithm in a programming language – simply explain it in sufficient detail to demonstrate that it runs in polynomial time. You do not need to compute the exact order of your algorithm.

QUESTION 4

Recall the *Partition Problem*: Given a set of integers $S = \{a_1, a_2, ..., a_n\}$, does S contain a subset that sums to exactly $\frac{\sum_i a_i}{2}$ (ie, half of the total sum) ?

We know that Partition is NP-Complete.

Consider this problem:

P_A : Given a set of integers T (which may contain duplicate values), can T be divided into 3 disjoint subsets that all sum to the same value?

For example, if $T = \{1, 1, 1, 3, 4, 5, 5, 7, 9\}$ then the answer to P_A is "Yes" because T can be divided into $\{1, 1, 1, 4, 5\}, \{5, 7\}, \{3, 9\}$ each of which sums to 12.

(a) [5 marks] Prove that P_A is in the class NP

(b) [10 marks] Prove that $Partition \propto P_A$